KINETIC THEORY OF THE UPPER ATMOSPHERE

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Abstract

From a system of Boltzmann kinetic equations for a gas mixture subjected to photon radiation, we formulate generalized hydrodynamics equations which include diffusion and thermodiffusion phenomena, chemical reactions and magnetohydrodynamic processes. The generalized hydrodynamic system is used in the analysis of upper atmospheric properties.

The book is offered for use by research specialists working in the field of atmosphere physics and cosmos, by graduate and under-graduate students specializing in physics and meteorology.

Foreword

The term "upper layers of the atmosphere" is not a very definite concept. The lower border of the region is determined for all practical purposes by the character of the process analyzed. When the sun radiation has a direct, immediate effect upon the atmosphere in a given region, in terms of the processes induced, becomes an upper atmosphere. The readers are referred to the introduction for the explanation of the last characteristic.

According to the criterion indicated, this book contains the entire

thermosphere and parts of the mesosphere as comprising the upper atmosphere.

An extensive factual material has been collected and numerous theo-
retical assumptions have been developed in regards to the properties of the
atmosphere at high altitudes. At that, the hydrodynamic method of describing
macroprocesses in the atmosphere has been already extended to include the
strato- and mesosphere, making possible the formulation of the meteorology
theory for these regions. We make an attempt, in the present book, to
expand the hydrodynamic description to include the thermosphere.

The upper atmosphere, from the point of view of physics presents a
multicomponent gas system affected by the external radiation flux. The
basic assumption of the theory is the idea that macroscopic properties of
this system are determined by the elementary processes involved, while the
adequate method of investigation is the kinetic equations method; Maxwell
and Boltzman have devised this method in their works. An attempt has been
made in the introduction to demonstrate that the kinetic equations method,
actually, is the most generalized method—if not the unique method—of all
analysis methods accepted at the present for the description of the upper
atmosphere. This statement is in accordance with the strong belief of the
authors and they try to defend it in their book. There appears the possi-
bility of building hydrodynamic equations by this method for the upper
atmosphere and the securing of a series of physical results without solving
kinetic equations. The present monograph is devoted to this problem.

Mathematical formulae resulting from the theory are unfortunately
too cumbersome and do not possess the desired finesse; this is the conse-
quence of the endeavor to bring the equations to their final results. In
this connection the authors may be treated as being old fashioned but

their excuse would be in the fact that their book is considered as a work which pursues primarily practical, not theoretical aims. Yet, it is possible that a diagram method could have been worked out, analogous to the Feinman method applied in quantum mechanics.

The book does not exhaust the entire problem, by any means, and is actually only the first step in this new area of physics of the upper atmosphere. The realization of the planned program will require much time and efforts and the authors will be happy if their work results in interest in the problems described and in the method for their solution as offered by the authors. Chapters 1 and 4 contain the preliminary material which should be the base for work in the direction indicated.

Giving a short characteristic of the book content we may state that chapters 1 and 2 are devoted to the method of kinetic equations and the formulation of generalized equation of hydrodynamics in the upper atmosphere. Chapter 3 presents the physical effect of the latter. Chapter 4 contains a summary of data which constitute the experimental basis of the theory.

The authors express their gratitude to the staff members of the section of physics of the high layers of the atmosphere of the Central Aerological Observatory; the discussions carried on were extremely useful in understanding many questions, and also to professor K.S. Shifrin for his valuable suggestions made at the time the monograph was readied for print.

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BASIC SYMBOLS

Pressure

p^α, p^β — partial pressures of components

$p = p^\alpha + p^\beta$

p^α_{jk}, p^β_{jk} — viscous stress tensors

$$p_{jk} = p^\alpha_{jk} + p^\beta_{jk}, \quad P_{jk} = p_{jk} - \frac{1}{3}\delta_{jk}p_{ll} \qquad P = p + \frac{p_{ll}}{3}$$

Densities

ρ_α, ρ_β — partial densities of components $\qquad \rho = \rho_\alpha + \rho_\beta$

n_α, n_β — densities of particles number $\qquad n = n_\alpha + n_\beta$

$$\Delta_\alpha = \frac{n_\alpha}{n}, \quad \Delta_\beta = \frac{n_\beta}{n}$$

n_φ — density of photon numbers

Velocities

$J_{\alpha_i}(\Phi^{(n)}_{ijk\ldots})$ — integrals in relation to moments of the n-order

u_α, u_β — macroscopic velocities of components

$$W_0 = k_\alpha u_\alpha + k_\beta u_\beta$$
$$V_0 = u_\beta - u_\alpha$$

$$W'_0 = k_\alpha u'_\alpha + k_\beta u'_\beta$$
$$V'_0 = V_0$$
$$u'_\alpha = -\frac{\rho_\beta V_0}{\rho_\alpha + \rho_\beta}, \quad u'_\beta = \frac{\rho_\alpha V_0}{\rho_\alpha + \rho_\beta}$$

Temperatures

T_α, T_β — components temperature

$$T_0 = \frac{m_\alpha T_\beta + m_\beta T_\alpha}{m_\alpha + m_\beta}$$

$$(mT) = m_\alpha T_\beta + m_\beta T_\alpha$$

$$k_\alpha = \sqrt{\frac{m_\alpha T_\beta}{m_\beta T_\alpha}}, \quad k_\beta = \sqrt{\frac{m_\beta T_\alpha}{m_\alpha T_\beta}}$$

Other characteristics

S^α, S^β —| vectors of components thermal flux

$$q = \frac{S^\alpha + S^\beta}{2}$$ vector of a full thermal flux

Microscopic characteristics of state

v_0 —| single vector in a given direction

v —| random single vector

$\bar{\sigma}_{\alpha\alpha}$, $\bar{\sigma}_{\alpha\beta}$ —| cross-sections of chemical reactions

$\sigma_{\alpha\alpha}$, $\sigma_{\alpha\beta}$ —| cross-sections of elastic collisions

d —| diameter of particle

$$\sigma_{\alpha\beta} = \frac{1}{4}(d_\alpha + d_\beta)^2$$

$d\Omega$ —| element of solid angle

m_α, m_β —| particle masses

$$(m) = m_\alpha + m_\beta$$

$$M = \frac{2 m_\alpha m_\beta}{m_\alpha + m_\beta}$$

f_α, f_β —| non-equilibrium distribution functions

$f_{0\alpha}$, $f_{0\beta}$ —| Maxwell equilibrium distribution functions

S_α^β, S_β^α —| integrals of elastic impacts

R_α^γ, R_γ^α —| integrals of reactions

v_α, v_β —| full velocities of alpha and beta particles

ξ_α, ξ_β —| thermal velocities of alpha and beta particles

$\Phi^{(n)}_{ijk...}$ —| linear combinations $v_i v_j v_k...$,| determined by formulae (1.5.1.) —(1.5.4.)

$M^{(n)}_{ijk...}$ —| moment of the n - order

p_α, p_β —| particle alpha and beta pulses

p_φ —| photon pulse

Chemical components

α, β, $\varkappa$, λ, μ —| gas mixture components

ι —|ions, ε —|electrons

Macroscopic ratios

η_1^α, η_2^β, η_1^β, η_2^α, η —| gas misture viscosity ratio

λ_1^α, λ_2^β, λ_1^β, λ_2^α, λ —| gas mixture heat conductivity, ratio

α_T —|thermal diffusion coefficient

D —|diffusion coefficient

Other symbols

c_0 —| light velocity

E —|electric field voltage (intensity)

H —|magnetic field intensity

F —|mass external force

e —|electron charge

g —|gravitational acceleration

δ_{jk} —|Kroneker symbol

I —|radiation intensity

Φ —|photon flux

Numbering formula and references

The numbering of formulae is independent for each individual paragraph (1, 2, 3, and so on).

The following rule has been accepted for references to formulae:

1) when the formula is contained in a given paragraph, the reference is limited to the ordinal number of the formula in the given paragraph, for instance (14);

2) when the formula is contained in a different paragraph of the same chapter the reference includes two numbers: first the number of the paragraph of the chapter then the ordinal number of the formula in the paragraph indicated, as for instance: (3.18);

3) when the formula is contained in a different chapter, the reference contains three numbers: first the number of the chapter, then the number of the paragraph and, finally the ordinal number of the formula as it appears in the paragraph, for instance: (2.4.8).

INTRODUCTION

First of all, we should probably give an explanation for the title of the book. The expression "upper atmosphere" has attained in the last years in the Russian scientific literature a well-understood context and means the upper atmosphere layers. In this book the lower border of the upper atmosphere is taken as being situated at the place where the gravitational-diffusive distribution of gases begins to play an important part. On the other hand, it has been proven by means of experiments that contrary to the properties displayed by lower layers, the characteristics of the upper atmosphere are, basically etermined by the direct effect of the sun.

Therefore, the upper atmosphere should be considered as a medium presenting

a non-insulated gas system affected by the external photon and corpuscular

radiation and, on the whole, not in a state of thermodynamic equilibrium.

It seems natural to apply the methods of the kinetic theory of gases

on the Boltzman equation to a theoretical analysis of this type of medium.

It is well known that the basis of kinetic equations and methods of their

solving contain a series of intricate questions, which are not, however,

considered in this book. In other words, this is not a book on the theory

of kinetic equations, but only on the application of this theory to a definite

physical problem. Following this path--we consider the possibility of the

simplest and physically proven method of obtaining certain important con-

clusions on the state of upper atmosphere layers, solving the mechanisms

of elementary processes in reactions taking place in the gas medium as

affected by photon and corpuscular radiation.

The subsequent investigation is therefore based on the Boltzman

equation system for gas mixtures, with collision integrals and reaction

integrals on the right. In order to solve the system the method of moments

is used in a form slightly different from the one used by Grade. Boltzman

equations are treated concretely in reactions about which we may assume

that they play the main part in the formation of the state of the atmosphere

at high latidues--predominantly in the thermosphere and, partially, in the

mesosphere.

As for Boltzman's equation derivation and the description of its

characteristics the questions may be studied in the books of A. Sommerfeld

"Thermodynamics and Statistical Physics" (Moscow, 1955), S. Chapman, T.

Cowling "Mathematical Theory of Non-homogeneous Gases" (M. 1960). A

two-component gas is considered in the latter book. General mention on chemical reactions may be found in the book by Geo. Hirshfeld, Ch. Curtiss and P. Byrd "Molecular theory of gases and liquids" (M. 1961).

The equations received in the monograph are not solved extensively, in other words, the computation of non-equilibrium distribution fuctions in its explicit aspect is not performed, as the problem is reduced to the construction of generalized hydrodynamics equations for the upper atmosphere in the so-called thirteen-moment approximation.

Maxwell already has demonstrated the possibility of obtaining a hydrodynamics system of equations from the kinetic equations of Boltzman for the simple gas. Further on, the manner of solving of Boltzman equation by reducing to hydrodynamics equations has been considered in the works of Gilbert, Enscog and Chapman, Barnett, Grade. The methods they suggested have their positive and negative sides. These methods constitute a considerable independent chapter in the kinetic gas theory. The most efficient method for our aims of investigating the properties of the upper atmosphere appears to be the method suggested by Grade and which is avriant of the general method of moments. It is shortly described in paragraph 4 - chapter 1.

The conventional hydrodynamics are formulated in the approximation of one gas without considering the effects of diffusion and thermal diffusion. It does not account for chemical transformations of the molecules, dissociation, recombination, excitation, etc. Yet in the upper atmosphere the latter processes, as they are induced by solar and cosmic radiation sometimes determine the properties and the behaviour of gases, while the diffusion phenomena play definitely a very considerable part. Methods used for the

analysis of the properties of the upper atmosphere are characterized by a
great variety and dissociation. We see here photochemistry and hydro-
dynamics the electromagnetic theory, theory of electron and ion collisions,
etc. The absence of one common method results in the fact that even com-
paratively simple processes in the upper atmosphere sometimes can be con-
sidered only qualitatively and separately from other processes. This situ-
ation may easily be proven by using, as an example, the theory on gravita-
tional-diffusive distribution of gases. The theory of this process suggested
by Nicolat is based on formulae obtained for the binary mixture of gases.
Yet, even the area of the atmosphere adjoining 100 km should be considered
as a system consisting of three components and over. In addition, charged
components are present in the upper atmosphere; these components are actually
electrons and ions affected by magnetic and electric fields. The electrons
in the Earth ionosphere have a high capacity for moving and, as shown by
recent investigations, have, in some areas of the ionosphere a higher degree
of temperature than the other components in the atmosphere. Therefore, even
at a small concentration the electrons play an important part in determining
heat conditions in the upper atmosphere. The part played by electrons is
particularly important in heat conductivity along the lines of force of the
earth magnetic field. Therefore, hydrodynamics of the upper atmosphere
must take into account all the above factors listed and should generalize
in this sense.

It is possible, of course, to attempt to formulate the generalized
hydrodynamic equation for the upper atmosphere in a phenomenological manner,
but there is then the possibility of overlooking many important processes
and a great many empirical coefficients will then have to be introduced.

Therefore, the derivation of generalized hydrodynamical equztions from the Boltzman equation system, which carries the mechanisms of processes subject to calculation seems to be the most natural and physically justifiable.

The system of correlated hydrodynamic equations taking into account the basic reactions which take place in the upper atmosphere has the following limitations: it is devised only for mechanisms relating to paired collisions. However, in spite of the rarefication of the gas forming the upper atmosphere there is a basis for the assumption that triple collisions may play a part as well. From the point of view of a developed theory the formation of the ozone layer cannot be considered, either (for the same reasons - the impossibility of taking into account the triple collisions). The question dealing with the extent of the existing limitation and the nature of the macroscopic properties is too complicated and can be fully clarified only on the basis of data obtained by experiments and observations.

In addition to limitations of the theory from below, there also exists a limitation depending upon the altitude above, which are dependent upon defects inherent to the Grade method. Grade himself gave two limitations for the applicability of his method: 1) a change in the flux velocity at a distance of a median length of a free path should not exceed 20% of the speed of sound and, 2) the change in absolute temperature of the gas over this distance should not exceed 20 % of its magnitude. The existing experimental data show that the first condition is always achieved in the atmosphere. The second condition can be recorded at:

$$\frac{5 \cdot 10^{-3}}{T P_{\text{мм рт. ст.}}} \frac{\partial T}{\partial z} \leqslant 0,2.$$

By substituting the values $T, \dfrac{\partial T}{\partial z}\Big|$ and p known from the CIRA,

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1965 atmosphere mockup, we see that at a 200 km altitude this condition is definitely attained. As the altitude increases the $\left.\dfrac{\partial T}{\partial z}\right|$ tends to zero, as absorbed solar energy, in other words, in proportion to the decrease in density. The length of the free path grows in inverse proportion to density. The left part of the above formula tends to a constant limit, considerably smaller than 0.2. This would provide a basis for the belief that the second condition of Grade is achieved in the entire atmosphere. At high altitudes the atmosphere is subjected to considerable 24 hour fluctuations. At those altitudes where the time between the particles collision may be commensurable with a 24 hour period the atmosphere may be in the non-equilibrium state. However, evaluations indicate that this non-equilibrium may originate only at altitudes considerably exceeding 800 km. Finally, as we analyze an open system, obsorbing radiation, we must admit that one more source of non-equilibrium exists. Detailed evaluations of this non-equilibrium have been made in paragraph 4 of chapter 3. We demonstrate that kinetic temperature and temperature defining the distribution according to pressure or density degree differ by 1% in relation to the last value of non-equilibrium at altitudes exceeding 400 to 500 km. Thus, the method used by us may be applied, without involving great errors in an altitude range of 100 to 500 km, as a minimum.

When formulating the system of generalized hydrodynamic equations the authors did not introduce the distribution function along the internal levels of particle freedom, as this fact did not affect the form of the equations obtained. If one and the same particle may be located at different levels of excitation of the internal levels of freedom this fact may be considered as a sufficient indication of their being isolated into

into individual components in the mixture. For all practical purposes, only two states were distinguished: the basic and the excitation state. However, this consideration of the component does not agree with the conventional understanding and, in addition, in a subsequent consideration of all excitation levels the number of components to be analyzed may increase to infinity. Therefore, a distribution fuction for internal levels of freedom has been introduced (paragraph 4, chapter 2). At that, we obtain $\frac{c_p}{c_v} = \frac{(i+2)}{i}$, where i is the total number of freedom levels instead of the 5/3 which should appear from the previous manner of consideration. Simultaneously, there is a relaxation form in the energy exchange between the internal and the progressive freedom levels.

Many important physical consequences may be obtained when using hydrodynamic equations even when they are not subsequently solved. The book deals extensively with these consequences and it appears that they open anentire series of new situations relating to the understanding of processes taking place in the upper stmosphere and to conditions of formations of its state depending upon the affect of radiation.

In order to establish fields of thermodynamic parameters and velocities of macroscopic motions using the solutions of hydrodynamic equations it is necessary to formulate the border and initial conditions —which is a question presenting an independent problem in itself, but, in addition we should also know the cross-section of reactions. In spite of the extensive material of experimental and theoretical investigations collected on the last problem, it appears that it is still insufficient for a complete understanding of the planned program and that a considerable amount of work has to be performed in this direction. At that, it is very important to have

a very distinct concept of the level and volume of information relating to the cross-sections of basic reactions. These data which will serve as basic material for the continued investigationsin the direction planned are listed in the last chapter.

The schematic method of the theory reported above permits to determine concrete problems on the computation of individual mockups of thermo- and mesospheres and prepares the way for the problem of forecasting on the state of these areas in connection with solar activity and certain other factors. The other possibility consists in he investigation of connections existing between various stratified layers of the atmosphere.

BOLTZMAN"S KINETIC EQUATIONS SYSTEM AND GRADE"S METHOD OF SOLUTION

1. <u>General aspect of the kinetic equations system of Boltzman for a</u>

<u>composite gas compound in the presence of reactions</u>

We shall start with the basic distribution function $f_\alpha(\mathbf{r}, \mathbf{v}_\alpha, t)$ of

dynamic states of one particle in a six-dimensional phase space of the x_i

coordinates and $v_{\alpha i}$ velocities (1, 2, 3). The x_i and $v_{\alpha i}$ magnitudes are

components of the r vector which determines the location of the particle

and the vector v_α which determines its velocity at a time moment t.

In a non-equilibrium gas mixture made up of n components, when the

interactions between the particles are limited to paired collisions which

do not result in a transformation of their nature the change f_a is described

by the system of Boltzman's kinetic equations

$$Df_\alpha = \sum_\beta S_\alpha^\beta,$$

(1)

where the summing up is distributed over all n components of the gas mixture

and the operator

$$D = \frac{\partial}{\partial t} + v_{\alpha i} \frac{\partial}{\partial x_i} + \frac{F_{\alpha i}}{m_\alpha} \frac{\partial}{\partial v_{\alpha i}},$$

(2)

while the force F_α with components $F_{\alpha i}$ affecting the particle is a function

of the point and the time.

The collision integral in the right part (1)

$$S_\alpha^\beta = \frac{1}{m_\beta} \iint \sigma_{\alpha\beta} |\mathbf{V}_{\beta\alpha} \cdot \mathbf{v}| (f'_\alpha f'_\beta - f_\alpha f_\beta) d\Omega \, d\mathbf{v}_\beta$$

(3)

may be considered as an operator transforming the dynamic state of the parti-

cles due to the occurring paired collisions. The shaded and unshaded pro-

duct of distribution functions have a meaning determined by the formulae:

$$f'_\alpha f'_\beta = f_\alpha \left(\mathbf{r},\ \mathbf{v}'_\alpha,\ t\right) f_\beta \left(\mathbf{r},\ \mathbf{v}'_\beta,\ t\right), \tag{4}$$

$$f_\alpha f_\beta = f_\alpha \left(\mathbf{r},\ \mathbf{v}_\alpha,\ t\right) f_\beta \left(\mathbf{r},\ \mathbf{v}_\beta,\ t\right), \tag{5}$$

while the connection between the shaded and unshaded velocities is given by the formula:

$$\mathbf{v}'_\alpha = \mathbf{v}_\alpha + \left(\mathbf{V}_{\beta\alpha} \cdot \mathbf{v}\right) \mathbf{v}, \tag{6}$$

$$\mathbf{v}'_\beta = \mathbf{v}_\beta - \left(\mathbf{V}_{\beta\alpha} \cdot \mathbf{v}\right) \mathbf{v}, \tag{7}$$

where

$$\mathbf{V}_{\beta\alpha} = \mathbf{v}_\beta - \mathbf{v}_\alpha \tag{8}$$

is the relative velocity of particles. The collision integral S^β_α includes the difference in operators containing products $f'_\alpha f'_\beta$ and $f_\alpha f_\beta$; the first and second correspond respectively to the transformation of dynamic states leading to an increase and decrease in the number of particles in time in a state determined by the volume element $drdv_\alpha$ near the point (r, v) of the phase space; v is the unit vector of the normal to the effective differential cross-section of the collision $\sigma_{\alpha\beta}$ in the element of the solid $d\Omega$ angle ϑ coinciding with the line of center of the colliding particles,

is the collision angle between v and the direction of the incident particle,

$$d\Omega = \sin\vartheta\, d\vartheta\, d\varphi, \tag{9}$$

where φ is the aximuth $(0 \leqslant \varphi \leqslant 2\pi)$.

Function f_α is rated for a mass density p_α, in such a way that

$$\int f_\alpha\, d\mathbf{v}_\alpha = p_\alpha. \tag{10}$$

In the simplest case of the one-component gas we should rate n = 1 in the above formulae. The system (1) then will become a single integro-differential equation of Boltzman for a single distribution function.

In a gas system affected by external effects many various reactions may take place, those inherent to the system itself at a given thermodynamic state as well as those excited by external agents, for instance, by photon or corpuscular flux. Such reactions are either positive or negative sources of a modified function f_α depending upon the fact whether they lead to the appearance or disappearance of the alpha particles.

The Boltzman equations system, which takes into account the reactions could formally be expressed in the following form:

$$Df_\alpha = \sum_\beta S_\alpha^\beta + \sum_\gamma R_\alpha^\gamma + \sum_\gamma R_\gamma^\alpha \tag{11}$$

where R_γ^α and R_α^γ are operators of separate processes of the (R_γ^α) which cause the appearance of (R_α^γ) and the disappearance of $(R \quad)$ of alpha particles in the phase volume $drdv_\alpha$ near the point (r, v_α).

In recording the concrete expressions for R_γ^α and R_α^γ as a symbol of the process we shall write the symbol of the initial products of the γ reaction.

Let us remark, at this point that the form in which the Boltzman's equation system is recorded (11) is based on the assumption that the reactions are independent. In fact, only with this assumption the right parts of the equations may be presented as a sum of operators of individual processes.

Expressions R_γ^α and R_α^γ will be in the future called the reaction integrals.

The value of the distribution functions is taken in point (r, v, t) everywhere and the (drdv) is the phase space element considered. However, in the future, for the sake of abbreviation, we shall delete the variables t and r in functional recordings and in recording ppoints and elements of

the phase space while concentrating our attention on the velocity space.

2. Mono-molecular and bi-molecular reactions

Inasmuch as the gas density in the upper atmosphere is sufficiently low the processes occurring within, can, basically, be regarded as mono- and bi-molecular.

1. Mono-molecular reaction

The monomolecular reaction of the dissociation of one particle into two

$$\mu^* \rightarrow \alpha + \beta \tag{1}$$

is described by the relations

$$\frac{dN_{\mu^*}}{dt} = -\frac{N_{\mu^*}}{\tau}, \tag{1}$$

$$\frac{dN_\alpha}{dt} = \frac{dN_\beta}{dt} = \frac{N_{\mu^*}}{\tau}, \tag{2}$$

where $N_{\mu^*}, N_\alpha, N_\beta$ — are the corresponding numerical densities of particles and τ is the life time of a dissociating particle not dependent upon its velocity. This conformity to the principle is justifiable for those particles which possess around point (v_{μ^*}) a velocity contained in the phase volume element dv_{μ^*}.

Then, taking into account the rate fixing (1.10), we will probably have

$$R_{\mu^*}^{\mu^*} = -\frac{1}{\tau} f_{\mu^*}(v_{\mu^*}) \tag{3}$$

—which is a general expression for the operator of the dissociation of the μ^* particle into two.

The decay of the μ^* particles leads to the appearance of alpha particles in such a manner that the velocity of their origination, according to (2) must be proportionate with the number of those μ^* particles which

possess dynamic states, situated in such a volume element $dv'_{\mu*}$, that after

decay the alpha particles acquire the velocity contained in dv_α.

The connection between the pulses of the particles formed and the

pulse of the initial particle is given by the law of pulse and energy

conservation.

If we are to consider the $\mu*$ particle pulse as given, then, out of

the four conservation laws we must determine six unknown pulse components of

the alpha and beta particles, consequently the pulse components of the latter

will depend upon the pulse of the initial particle and two parameters.

It is a known fact that the unit vector k can be determined in a

one-digit manner by two angles (ϑ and φ) which can be considered as

being just such two parameters, in other words, record to functional con-

nections

$$v'_\alpha = Y_1 (v_{\mu*}, \ k),$$

(4)

$$v'_\beta = Y_2 (v_{\mu*}, \ k),$$

(5)

where the form of functions Y_1 and Y_2 is determined by the energy and pulse

conservation laws. Each direction of the vector k (ϑ, φ) can have the

probability $\dfrac{d\Omega}{4\pi}$, added to its value and then, on the basis of the above

consideration we shall obtain the expression for operators of the origina-

tion of alpha and beta particles.

(6)

$$R^\alpha_{\mu*} = \frac{1}{4\pi\tau} \int f_{\mu*}(v'_\alpha, \ k)\, d\Omega,$$

$$R^\beta_{\mu*} = \frac{1}{4\pi\tau} \int f_{\mu*}(v'_\beta, \ k)\, d\Omega.$$

(7)

When the particle decays into two equal particles, then as can be

seen from (4) and (5) the v'_α function will be two-digital, which will be

in full agreement with the existence of the last two operators.

2. Bimolecular reactions

As a result of the bimolecular reaction two colliding particles form a new system of one or several particles. For the reaction in the form

$$\alpha + \beta \rightarrow \varkappa + \lambda \qquad\qquad (II)$$

$$\frac{dN_\alpha}{dt} = -\bar{\sigma}_{\alpha\beta} \, | V_{\beta\alpha} | \, N_\alpha N_\beta , \qquad\qquad (8)$$

$$\frac{dN_\varkappa}{dt} = \bar{\sigma}_{\alpha\beta} \, | V_{\beta\alpha} | \, N_\alpha N_\beta , \qquad\qquad (9)$$

where $\bar{\sigma}_{\alpha\beta}$ is the differential section of the reaction (is opposed to the elastic cross-section of collision $\sigma_{\alpha\beta}$ this magnitude is marked with a dash over the section symbol). In the simplest case, where the cross-section of collision is expressed by a continuous function of any parameter (for instance, temperature), the cross-section of the reaction may be presented in the form of the product of the collision cross-section and the probability of the onset of the $\bar{\sigma}_{\alpha\beta}$ reaction in the presence of collision.

At this point the cross-section of the elastic collision would be equal to the product of the collision cross-section with 1 minus the probability bility of the reaction. However, the correctness of this assumption must be specifically investigated in each individual case, and, in addition, we shall see processes farther below, for which the considered suggestion is at the very least doubtful. In view of this, we introduce, from the very start, various symbols to indicate the cross-sections of elastic collisions and of the reaction cross-section, considering the first as being independent from the second.

Taking into consideration (8) and (9) and keeping in mind that the distribution functions are rated according to density, in analogy with (1.3) it is possible to write an equation for function $f_\varkappa$

$$dv_x \frac{1}{m_x} Df_x = \frac{1}{m_\alpha m_\beta} \iint \bar{\sigma}_{\alpha\beta} \left| V'_{\beta\alpha} \cdot \nu \right| f'_\alpha f'_\beta \, d\Omega \, dv'_\alpha \, dv'_\beta, \quad (10)$$

where the integration is assumed to be effected along $d\Omega \, dv'_\beta$.

The shaded relative velocity

$$V'_{\beta\alpha} = v'_\beta - v'_\alpha \qquad (11)$$

in accordance with formulae (1.6) and 1.7) means such a value at which, the particle x, receives, as a result of the reaction of the value of the velocity around v_x.

Inasmuch as

$$dv'_\alpha \, dv'_\beta = \frac{\partial \left(v'_\alpha, \, v'_\beta \right)}{\partial \left(v_x, \, v_\lambda \right)} dv_x \, dv_\lambda, \qquad (12)$$

the integral of the onset of the reaction of the particle takes the form:

$$R^x_{\alpha\beta} = \frac{m_x}{m_\alpha m_\beta} \iint \bar{\sigma}_{\alpha\beta} \left| V'_{\beta\alpha} \cdot \nu \right| f'_\alpha f'_\beta \, d\Omega \frac{\partial \left(v'_\alpha, \, v'_\beta \right)}{\partial \left(v_x, \, v_\lambda \right)} dv_\lambda. \quad (13)$$

The connection between v'_ϵ, v'_β, v_x and v_λ is given by formulae(A,8) in the A addendum, therefore the sub-integral expression is a function v_x and v_λ.

The remaining integrals of the reaction (II) have the form

$$R^{\alpha\beta}_\alpha = -\frac{1}{m_\beta} \iint \bar{\sigma}_{\alpha\beta} \left| V_{\beta\alpha} \cdot \nu \right| f_\alpha f_\beta \, d\Omega \, dv_\beta, \qquad (14)$$

$$R^\lambda_{\alpha\beta} = \frac{m_\lambda}{m_\alpha m_\beta} \iint \bar{\sigma}_{\alpha\beta} \left| V'_{\beta\alpha} \cdot \nu \right| f'_\alpha f'_\beta \, d\Omega \frac{\partial \left(v'_\alpha, \, v'_\beta \right)}{\partial \left(v_x, \, v_\lambda \right)} dv_x, \qquad (15)$$

$$R^{\alpha\beta}_\beta = -\frac{1}{m_\alpha} \iint \bar{\sigma}_{\alpha\beta} \left| V_{\beta\alpha} \cdot \nu \right| f_\alpha f_\beta \, d\Omega \, dv_\alpha. \qquad (16)$$

Paragraph 3. Integrals of reactions for some individual concrete
processes

The upper atmosphere presents a complex photochemical laboratory
where a multitude of reactions take place. Using, subsequently, the equa-
tions system (1.11) in the upper atmosphere analysis, we shall limit our-
selves to those reactions of which we know that they ⸱ play an important
part.

These are, in the first place, the reactions of photodissociation of
molecules and the recombination of atoms, photo-ionization and recombination
of electrons and ions, of excitation and spontaneous radiation, of attenua-
tion of the excitation during collisions and certain other ⸱ phenomena.
These reactions are, of course, insufficient for the description of all
properties of the atmosphere, but it can be hoped that they will serve as
a certain physical model of interaction of the solar radiation and the Earth
atmosphere, in the first approximation which actually determines the basic
properties of the upper atmosphere.

Let us now consider certain concrete reactions and record their in-
tegrals.

(1) Collisions of particles with different masses (3)

The collision of particles with different masses may be considered
as a bimolecular reaction

$$\alpha + \beta \rightarrow \alpha + \beta. \qquad (III)$$

Its particular feature is in the fact that the reaction integrals
can be united into one member $R_\alpha^{\gamma\beta} + R_{\alpha\beta}^\alpha = S_\alpha^\beta = \dfrac{1}{m_\beta} \iint \sigma_{\alpha\beta} |V_{\beta\alpha} \cdot \nu| \left(f_\alpha' f_\beta' - f_\alpha f_\beta \right) d\Omega \times$

$$\times dv_\beta. \qquad (1)$$

$S_\beta^\alpha|$ has the same form but the alpha and beta symbols exchange their places.
The connection between the particle velocities before and after the collision

is give by formulae (A.11) in the addendum A.

2) Reaction of particles excited by a photon (3)

This reaction belongs in the bimolecular reactions group although itth is somewhat extraordinary. The reaction formula has the form:

$$\mu + \varphi \rightarrow \mu^*. \qquad (IV)$$

Due to the participation of photons in the reaction it is expedient to consider the process in the pulse phase space $p(p_x, p_y, p_z)$ instead of in the phase space of velocities.

A reaction is achieved only in the case when the molecule excitation level corresponds to the sum of the photon energy and the kinetic energy of the molecule in their inertia center system.

The reaction integrals have the form

$$R_\mu^{\mu\varphi} = - \int\int \bar{\sigma}_{\mu\varphi} \mid c_0 \cos \vartheta \mid f_\mu f_\varphi \, d\Omega \, d\mathbf{p}_\varphi, \qquad (2)$$

$$R_{\mu\varphi}^{\mu^*} = \int\int \bar{\sigma}_{\mu\varphi} \mid c_0 \cos \vartheta \mid f'_\mu f'_\varphi \, d\Omega \, \frac{\partial \left(\mathbf{p}'_\mu \right)}{\partial \left(\mathbf{p}_{\mu^*} \right)} \, d\mathbf{p}_\varphi. \qquad (3)$$

Inasmuch as the photons do not possess a rest mass their pulse distribution functions are rated for the numerical density of photons. The relative velocity of a photon and a molecule equals the speed of light c_0, ϑ — is the angle formed by the direction v and the direction of the photon flight.

Let us explain the last equation. We have

$$d\mathbf{p}_{\mu^*} Df_{\mu^*} = \left[\int\int \bar{\sigma}_{\mu\varphi} \mid c_0 \cos \vartheta \mid f'_\mu f'_\varphi \, d\Omega \, d\mathbf{p}_\varphi \right] d\mathbf{p}'_\mu, \qquad (4)$$
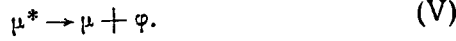
yet $\qquad \dfrac{d\mathbf{p}'_\mu}{d\mathbf{p}_{\mu^*}} = \dfrac{\partial \left(\mathbf{p}'_\mu \right)}{\partial \left(\mathbf{p}_{\mu^*} \right)}$, a $\mathbf{p}_{\mu^*} = \mathbf{p}'_\mu + \mathbf{p}_\varphi$ [формула (A. 12) (formula (A.12)

of addendum A). Wherefrom:

$$\frac{d\mathbf{p}'_\mu}{d\mathbf{p}_{\mu^*}} = 1, \quad d\mathbf{v}'_\mu = d\mathbf{v}_{\mu^*}. \qquad (5)$$

3) Reaction of a spontaneous radiation of an excited particle (3)

The reaction formula

$$\mu^* \to \mu + \varphi. \qquad\qquad (V)$$

The reaction of spontaneous radiation belongs to the monomolecular reactions group in the same manner as the reaction of dissociation of an activated molecule (integrals of reaction (2.3), (2.6), (2.7)].

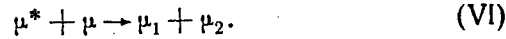The integrals of this reaction are in the form

$$R_{\mu^*}^{\mu^*} = -\frac{1}{4\pi\tau} \int f_{\mu^*}(\mathbf{p}_{\mu^*})\, d\Omega, \qquad (6)$$

$$R_{\mu^*}^{\mu} = \frac{1}{4\pi\tau} \int f_{\mu^*}(\mathbf{p}'_{\mu^*})\, d\Omega. \qquad (7)$$

The connection between the pulses of particles μ and μ^* is demonstrated in formulae (A.13) of the addendum A.

4) The reaction of excitation damping in collisions with a non-excited particle (3)

This is a bimolecular reaction which we shall record as follows:

$$\mu^* + \mu \to \mu_1 + \mu_2. \qquad\qquad (VI)$$

On the basis of paragraph 2 it is easy to obtain the expressions of the reaction integrals

$$R_{\mu}^{\mu\mu^*} = -\frac{1}{m} \iint \bar{\sigma}_{\mu\mu^*} |\, \mathbf{V}_{\mu^*\mu} \cdot \mathbf{v}|\, f_\mu f_{\mu^*} d\Omega\, d\mathbf{v}_{\mu^*}, \qquad (8)$$

$$R_{\mu^*}^{\mu\mu^*} = -\frac{1}{m} \iint \bar{\sigma}_{\mu\mu^*} |\, \mathbf{V}_{\mu^*\mu} \cdot \mathbf{v}|\, f_\mu f_{\mu^*} d\Omega\, d\mathbf{v}_{\mu}, \qquad (9)$$

$$R_{\mu\mu^*}^{\mu_1} = \frac{1}{m} \iint \bar{\sigma}_{\mu\mu^*} |\, \mathbf{V}'_{\mu^*\mu} \cdot \mathbf{v}|\, f'_\mu f'_{\mu^*} d\Omega\, d\mathbf{v}_{\mu_2} \frac{\partial(\mathbf{v}'_\mu,\, \mathbf{v}'_{\mu^*})}{\partial(\mathbf{v}_{\mu_1},\, \mathbf{v}_{\mu_2})}, \quad (10)$$

$$R_{\mu\mu^*}^{\mu_2} = \frac{1}{m} \iint \bar{\sigma}_{\mu\mu^*} |\, \mathbf{V}'_{\mu^*\mu} \cdot \mathbf{v}|\, f'_\mu f'_{\mu^*} d\Omega\, d\mathbf{v}_{\mu_i} \frac{\partial(\mathbf{v}'_\mu,\, \mathbf{v}'_{\mu^*})}{\partial(\mathbf{v}_{\mu_1},\, \mathbf{v}_{\mu_2})}. \quad (11)$$

We notice that actually particles μ_1 and μ_2 do not differ from mu and the corresponding members may be associated. The particle pulses before and after the reaction are connected by formulae (A.14) in addendum A.

5. Reaction of photodissociation (photoionization) (3)

The reaxtion expression

$$\mu + \varphi \to \alpha + \beta \qquad\qquad \text{(VII)}$$

(α, β - are symbols of atoms which originated as a result of the dissociation.

The integrals of the reaction will be recorded in the form of

$$R_\mu^{\mu\varphi} = - \iint \bar{\sigma}_{\mu\varphi} |c_0 \cos\vartheta| f_\mu f_\varphi \, d\Omega \, d\mathbf{p}_\varphi, \qquad (12)$$

$$R_{\mu\varphi}^\alpha = \frac{m_\alpha}{m} \iint \bar{\sigma}_{\mu\varphi} |c_0 \cos\vartheta| f_\mu' f_\varphi' \, d\Omega \, d\mathbf{p}_\beta \, \frac{\partial(\mathbf{p}_\mu', \mathbf{p}_\varphi')}{\partial(\mathbf{p}_\alpha, \mathbf{p}_\beta)}, \quad (13)$$

$$R_{\mu\varphi}^\beta = \frac{m_\beta}{m} \iint \bar{\sigma}_{\mu\varphi} |c_0 \cos\vartheta| f_\mu' f_\varphi' \, d\Omega \, d\mathbf{p}_\alpha \, \frac{\partial(\mathbf{p}_\mu', \mathbf{p}_\varphi')}{\partial(\mathbf{p}_\alpha, \mathbf{p}_\beta)}. \quad (14)$$

We have analogous expressions for the ions (ι) and the electrons (ε) where the symbols ι and ε are situated in the place of alpha and beta, respectively. The connection between the pulses of the particles before and after the reaction is expressed in formulae (A.18) and (A.19) of addendum A.

6) Reaction of recombination with photon radiation (3)

The reaction formula is

$$\alpha + \beta \to \mu + \varphi. \qquad\qquad \text{(VIII)}$$

The integrals of the reactions for the recombination of atoms have the following expression

$$R_\alpha^{\mu\varphi} = - \frac{1}{m_\beta} \iint \bar{\sigma}_{\alpha\beta} |\mathbf{V}_{\beta\alpha} \cdot \mathbf{v}| f_\alpha f_\beta \, d\Omega \, d\mathbf{p}_\beta, \qquad (15)$$

$$R_\beta^{\alpha\beta} = - \frac{1}{m_\alpha} \iint \bar{\sigma}_{\alpha\beta} |\mathbf{V}_{\beta\alpha} \cdot \mathbf{v}| f_\alpha f_\beta \, d\Omega \, d\mathbf{p}_\alpha, \qquad (16)$$

$$R_{\alpha\beta}^{\mu} = \frac{m}{m_\alpha m_\beta} \int \int \bar{\sigma}_{\alpha\beta} \, | \, V_{\beta\alpha}' \cdot \nu \, | \, f_\alpha' f_\beta' \, d\Omega \, d\mathbf{p}_\varphi \frac{\partial \left(\mathbf{p}_\alpha', \, \mathbf{p}_\beta' \right)}{\partial \left(\mathbf{p}_\varphi, \, \mathbf{p}_\mu \right)} . \quad (17)$$

Analogous expressions have been obtained for ions and electrons, with however, the difference that in the latter case the symbols ι and ε will replace the alpha and beta symbols respectively.

The particle pulses before and after the reactions are connected by the expressions (A.22) and (A.23) of addendum A.

Paragraph 4. Grade method of solution of a kinetic equation

Grade used the momentum method to solve the kinetic equation for a one-component gas; to this end he applied the Ermit polynomial expansion of the distribution function which can be expressed by the local Maxwell distribution function.

(1) In order to find the expansion factor for a non-equilibrium distribution function it is necessary to solve, without successive approximations the final system for hydrodynamics equations whose number is determined by the number of physical characteristics necessary for the description of the system. The presentation of the Grade method will be reported, following, basically, in the main features, Sommerfeld's (2).

Boltzman equation for a one-component gas is easily obtained from the system (1.1). In fact, we should establish that n = 1 in this case and the system then becomes one equation in which the symbol beta represents molecules of the same single component alpha. Therefore, we may omit the symbols alpha and beta in the kinetic equation and write

$$Df = S \qquad (1)$$

By deleting, for the same reasons the symbols y, m. and V for the collision integral we obtain:

$$S = \frac{1}{m} \iint \sigma \, | \mathbf{V} \cdot \mathbf{v} | \left(f' f'_\beta - f f_\beta \right) d\Omega \, d\mathbf{v}_\beta , \quad (2)$$

where $\frac{1}{m} f_\beta d\mathbf{v}_\beta$ — is the number of molecules in the dynamic state around

point $\mathbf{v}_{\beta}$.

Grade searched a solution for equ. (1) in the form of the distribu-

tion of function expansion with the generalized Ermit polynoms:

$$f = f_0 \sum_{n=0}^{\infty} \frac{1}{n!} \, b_{ijk}^{(n)} \ldots H_{ijk}^{(n)} \ldots , \quad (3)$$

where

$$H_{ijk}^{(n)} \ldots = e^{\frac{\mathbf{c}^2}{2}} \frac{\partial^n}{\partial c_i \partial c_j \partial c_k \ldots} \left(e^{-\frac{\mathbf{c}^2}{2}} \right) \quad (4)$$

—polynom in the three-dimensional space of components of the "reduced" velo-

city $$\mathbf{c} = \frac{\mathbf{v} - \mathbf{u}'}{\sqrt{\frac{kT'}{m}}} , \quad (5)$$

u' and T' are the parameters, k — is the Boltzman's constant and, finally

$$f_0 = \rho' \left(\frac{m}{2\pi kT'} \right)^{3/2} e^{-\frac{\mathbf{c}^2}{2}} \quad (6)$$

— is the solution of the equation (1) in a zero approximation, in other words,

the local Maxwell distribution function containing ρ', T' and u' as para-

meters.

The ρ', T' and u' magnitudes should be determined by means of charac-

teristics of the gas state.

These characteristics are the density ρ, the pressupre p and the gas

velocity u, which are determined by means of the non-equilibrium distribu-

tion function f:

$$\rho = \int f \, d\mathbf{v} ,$$
$$\rho \mathbf{u} = \int \mathbf{v} f \, d\mathbf{v} ,$$
$$p = \int \frac{(\mathbf{v} - \mathbf{u})^2}{3} f \, d\mathbf{v} .$$

In order to connect the parameters of the Maxwell distribution function with the characteristics of the state Grade established the following conditions:

$$\rho = \int f_0 \, d\mathbf{v}, \qquad (7)$$

$$\rho \mathbf{u} = \int \mathbf{v} f_0 \, d\mathbf{v}, \qquad (8)$$

$$p = \int \frac{(\mathbf{v} - \mathbf{u})^2}{3} f_0 \, d\mathbf{v}, \qquad (9)$$

$$p = \frac{\rho k T}{m}. \qquad (10)$$

After computations, we find that

$$\rho' = \rho,$$
$$\mathbf{u}' = \mathbf{u},$$
$$T' = \frac{pm}{k\rho}.$$

This selection of parameters of the local distribution function — as we shall see below — brings about certain limitations to factors $b_i^{(1)}$ and b_{ij} .

Let us remark that this selection of parameters is not the only possible one.

The expression (4) for the Ermit polynoms, may, evidently, be also presented in the form:

$$H_{ijk\ldots}^{(n)} = \frac{1}{f_0} \frac{\partial^n f_0}{\partial c_i \, \partial c_j \, \partial c_k \ldots}. \qquad (11)$$

The expansion (3) upon transition to physical components of heat velocity is then

$$\xi = \mathbf{v} - \mathbf{u} \qquad (12)$$

and introduction of new factors $a_{ijk}^{(n)} \ldots,$ connected with the previous formula

$$b_{ijk}^{(n)} \ldots \left(\frac{kT}{m}\right)^{n/2} = \frac{a_{ijk}^{(n)} \ldots}{\rho}, \qquad (13)$$

may be written in the form

$$f = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{a_{ijk}^{(n)} \ldots}{\rho} \frac{\partial^n f_0}{\partial \xi_i \partial \xi_j \partial \xi_k \ldots}. \qquad (14)$$

From the start we may establish that $a^{(0)} = \rho$. The expansion (14) then, starts with the Maxwell function of equilibrium gas.

Let us introduce the moment concept . The moment of the n order is the magnitude

$$M_{ijk}^{(n)} \ldots = \int (v_i v_j v_k \ldots) f \, d\mathbf{v}. \qquad (15)$$

The first moments have a definite physical sense (value)

$$M^{(0)} = \int f \, d\mathbf{v} = \rho, \qquad (16)$$

$$M_i^{(1)} = \int v_i f \, d\mathbf{v} = \rho u_i, \qquad (17)$$

$$M_{ii}^{(2)} = \int \mathbf{v}^2 f \, d\mathbf{v} = \int \{(\mathbf{v} - \mathbf{u})^2 + 2(\mathbf{v} - \mathbf{u}) \cdot \mathbf{u} + \mathbf{u}^2\} f \, d\mathbf{v}. \qquad (18)$$

Using (9),(16) and (17) we obtain

$$M_{ii}^{(2)} = 3p + \rho u^2. \qquad (19)$$

In the general form the moment of the second order can be written as

$$M_{ij}^{(2)} = \int v_i v_j f \, d\mathbf{v} = \int (u_i + \xi_i)(u_j + \xi_j) f \, d\xi =$$
$$= \rho u_i u_j + \int \xi_i \xi_j f \, d\xi. \qquad (20)$$

Let us present

$$\int \xi_i \xi_j f \, d\xi = P_{ij}.$$

Then

$$M_{ij}^{(2)} = \rho u_i u_j + P_{ij}. \qquad (21)$$

Tensor

$$p_{ij} = P_{ij} - p\delta_{ij}, \qquad (22)$$

as we shall see later is the tensor of viscous tensions.

It is clear from (9) that

$$p_{ii} = 0. \tag{23}$$

The third-order moment $M_{ijk}^{(3)}$ we shall report in the form

$$M_{ijk}^{(3)} = \int v_i v_j v_k f \, d\mathbf{v} = \int (u_i + \xi_i)(u_j + \xi_j)(u_k + \xi_k) f \, d\xi =$$
$$= \rho u_i u_j u_k + u_i P_{jk} + u_j P_{ik} + u_k P_{ij} + \int \xi_i \xi_j \xi_k f \, d\xi. \tag{24}$$

The last integral shall be designated as $S_{ijk}.$ It is a tensor of the third rank which characterizes heat transfer in the gas.

We establish a connection between the factors $a_{ijk...}^{(n)}$ in the distribution function expansion (14) and the moments determined above.

We analyze the first order moment

$$M_i^{(1)} = \int v_i f \, d\mathbf{v} = \int (u_i + \xi_i)\left(f_0 + \frac{a_n^{(1)}}{1!\rho}\frac{\partial f}{\partial \xi_n} + \right.$$
$$\left. + \frac{a_{mn}^{(2)}}{2!\rho}\frac{\partial^2 f_0}{\partial \xi_m \partial \xi_n} + \ldots\right) d\xi. \tag{25}$$

Before we compute the presented integral we sould prove the auxiliary theorem: integral

$$I = \int (\xi_i \xi_j \xi_k \ldots) \frac{\partial^l f_0}{\partial \xi_m \ldots \partial \xi_n} \, d\xi = 0,$$

when l is larger even by one unit than the number of the co-factors in the parenthesis N.

In fact $$I = \int d\xi_1 \, d\xi_2 \int (\xi_i \xi_j \xi_k \ldots) \, d \frac{\partial^{l-1} f_0}{\underbrace{\partial \xi_m \ldots \partial \xi_n}_{l-1}}.$$

Let us take this integral and analyze it N times by parts. We obtain

$$I = \sum_{k=1}^{N} \int d\xi_1 \, d\xi_2 \, (-1)^k \left[\frac{\partial^k (\xi_i \xi_j \xi_k \ldots)}{\underbrace{\partial \xi_m \ldots \partial \xi_n}_{k}} \frac{\partial^{l-k} f_0}{\underbrace{\partial \xi_m \ldots \partial \xi_n}_{l-k}}\right]_{-\infty}^{+\infty} \pm$$
$$\pm \int \frac{\partial^N (\xi_i \xi_j \xi_k \ldots)}{\underbrace{\partial \xi_m \ldots \partial \xi_n}_{N}} \cdot \frac{\partial^{l-N} f_0}{\underbrace{\partial \xi_m \ldots \partial \xi_n}_{l-N}} \, d\xi.$$

All members of the sum become zero, as $\xi_i = \pm\infty$ f_0 becomes a zero together with all its derivatives. The last integral will also become zero if $1 \quad N$, for the same reason. Utilizing this theorem, we write (25) as

$$M_i^{(1)} = \int u_i f \, d\xi + \int \xi_i \left(f_0 + \frac{a_n^{(1)}}{1! \rho} \frac{\partial f_0}{\partial \xi_n} \right) d\xi.$$

The first integral, according to (16) = pu. Let us compute the second integral

$$\int \xi_i \left(f_0 + \frac{a_n^{(1)}}{1! \rho} \frac{\partial f_0}{\partial \xi_n} \right) d\xi = \int d\xi_1 \, d\xi_2 \frac{a_n^{(1)}}{1! \rho} \int \xi_i \, df_0 =$$

$$= \int d\xi_1 \, d\xi_2 \frac{a_n^{(1)}}{1! \rho} \left[\xi_i f_0 \Big|_{-\infty}^{+\infty} - \int \frac{\partial \xi_i}{\partial \xi_n} f_0 \, d\xi_n \right] =$$

$$= -\frac{a_n^{(1)}}{1! \rho} \delta_{in} \int f \, d\xi = -a_i^{(1)}.$$

We obtain finally

$$M_i^{(1)} = \rho u_i - a_i^{(1)},$$

yet, according to (17) $M_i^{(1)} = \rho u_i$. It follows from this that $a_i^{(1)} = 0.$ (26)

Let us consider the moment of the second order

$$M_{ij}^{(2)} = \int v_i v_j f \, dv = \int (u_i + \xi_i)(u_j + \xi_j) \times$$

$$\times \left(f_0 + \frac{a_{mn}^{(2)}}{2! \rho} \frac{\partial^2 f_0}{\partial \xi_m \partial \xi_n} + \ldots \right) d\xi.$$

Here, the condition (26) has been utilized. By applying the theorem, proven above (9) and (16), we obtain

$$M_{ij}^{(2)} = \rho u_i u_j + p\delta_{ij} + \int \xi_i \xi_j \frac{a_{mn}^{(2)}}{2! \rho} \cdot \frac{\partial^2 f_0}{\partial \xi_m \partial \xi_n} \, d\xi.$$

Let us compute the last integral:

$$\int \xi_i \xi_j \frac{a_{mn}^{(2)}}{2! \rho} \frac{\partial^2 f_0}{\partial \xi_m \partial \xi_n} \, d\xi = \frac{a_{mn}^{(2)}}{2\rho} \int \frac{\partial^2 \xi_i \xi_j}{\partial \xi_n \partial \xi_n} f_0 \, d\xi,$$

but

$$\frac{\partial^2 \xi_i \xi_j}{\partial \xi_m \partial \xi_n} = \frac{\partial}{\partial \xi_m} (\xi_i \delta_{jn} + \xi_j \delta_{in}) = \delta_{jn} \delta_{im} + \delta_{jm} \delta_{in}.$$

It follows from that $M_{ij}^{(2)} = \rho u_i u_j + p\delta_{ij} + a_{ij}^{(2)}.$

By comparing the expression with (21) we obtain

$$a_{ij}^{(2)} = p_{ij}. \tag{27}$$

For the third order moment we have

$$M_{ijk}^{(3)} = \int v_i v_j v_k f \, dv = \int (u_i + \xi_i)(u_j + \xi_j)(u_k + \xi_k) \times$$

$$\times \left(f_0 + \frac{p_{mn}}{2\rho} \frac{\partial^2 f_0}{\partial \xi_m \partial \xi_n} + \frac{a_{mnl}^{(3)}}{6\rho} \frac{\partial^3 f_0}{\partial \xi_m \partial \xi_n \partial \xi_l} + \ldots \right) d\xi.$$

More cumbersome computations analogous to the above result in

$$M_{ijk}^{(3)} = \rho u_i u_j u_k + u_i P_{jk} + u_j P_{ik} + u_k P_{ij} - a_{ijk}^{(3)}.$$

Comparing this expression with (24) we obtain

$$a_{ijk}^{(3)} = -S_{ijk}. \qquad (28)$$

Thus we have established

$$
\begin{aligned}
a^{(0)} &= \rho, \\
a_i^{(1)} &= 0, \\
a_{ij}^{(2)} &= p_{ij}, \\
a_{ijk}^{(3)} &= -S_{ijk}.
\end{aligned}
\qquad (29)
$$

Inasmuch as the physical sense of the moments is unveiled by utilizing

conditions (7) − (10) which determine the selection of parameters of the local

distribution function, the selection of these conditions also involves cer-

tain limitations on factors $a_i^{(1)}$ and the trace of tensor p_{ij}. In particular,

the Grade scheme carries $a_i^{(1)} = 0$ equal to 0 and P_u equals zero.

Now expansion (14) for f may be re-written in the form of

$$f = f_0 + \frac{p_{ij}}{2\rho} \frac{\partial^2 f_0}{\partial \xi_i \partial \xi_j} - \frac{S_{ijk}}{6\rho} \frac{\partial^3 f_0}{\partial \xi_i \partial \xi_j \partial \xi_k} + \cdots \qquad (30)$$

In the future we shall need the value of the integral

$$\widehat{M}_{ijkl}^{(4)} = \int \xi_i \xi_j \xi_k \xi_l \left\{ f_0 + \frac{p_{mn}}{2\rho} \frac{\partial^2 f_0}{\partial \xi_m \partial \xi_n} - \frac{S_{mnk}}{6\rho} \frac{\partial^3 f_0}{\partial \xi_m \partial \xi_n \partial \xi_k} \right\} d\xi.$$

$$(31)$$

The third member in parenthesis transforms the corresponding integral

into zero in view of the odd value of the sub-integral function. The second

integral, when analyzed by parts is transformed into

$$\frac{p_{mn}}{2\rho} \int \frac{\partial^2 \xi_i \xi_j \xi_k \xi_l}{\partial \xi_m \partial \xi_n} f_0 \, d\xi,$$

while the further computations will bring it to the following form:

$$\frac{1}{\rho} \int [\xi_i \xi_j p_{kl} + \xi_i \xi_k p_{jl} + \xi_i \xi_l p_{jk} + \xi_j \xi_k p_{il} + \xi_j \xi_l p_{ik} + \xi_k \xi_l p_{ij}] \times$$

$$\times f_0 \, d\xi = \frac{\rho}{\rho} \{ p_{kl} \delta_{ij} + p_{jl} \delta_{ik} + p_{jk} \delta_{il} + p_{il} \delta_{jk} + p_{ik} \delta_{jl} + p_{ij} \delta_{kl} \}.$$

In the first integral, in the product $\xi_i\xi_j\xi_k\xi_l$ at least two indexes coincide. Let us assume that $l=i$. This integral then equals

$$\frac{1}{2}\int d\xi_1\,d\xi_2\int \xi_i\xi_j\xi_k f_0\,d\xi_i^2 = \frac{1}{2}\int d\xi_1\,d\xi_2\left\{\left[\xi_i\xi_j\xi_k\int f_0\,d\xi_i^2\,\Big|_{-\infty}^{+\infty}\right] - \int\frac{\partial\xi_i\xi_j\xi_k}{\partial\xi_l}\left[\int f_0\,d\xi_i^2\right]d\xi_l\right\}.$$

The first member in parenthesis produces zero due to the odd value of the function $\xi_i\xi_j\xi_k f_0\,d\xi_l^2$, the second is reduced to the form

$$p\frac{kT}{m}(\delta_{il}\delta_{jk}+\delta_{ik}\delta_{jl}+\delta_{kl}\delta_{ij})$$

Therefore, in the final result we have

$$\widehat{M}_{ljkl}^{(4)}=p\frac{kT}{m}(\delta_{il}\delta_{jk}+\delta_{ik}\delta_{jl}+\delta_{kl}\delta_{ij})+\frac{kT}{m}(p_{kl}\delta_{ij}+p_{jl}\delta_{ik}+$$
$$+p_{jk}\delta_{il}+p_{il}\delta_{jk}+p_{ik}\delta_{jl}+p_{ij}\delta_{kl}). \qquad (32)$$